''Two-stage'' perturbation theory for bandwidth-limited amplification of optical solitons near the zero-dispersion point

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We show that soliton propagation in transmission lines where bandwidth-limited amplification, nonlinear gain or loss, and a strong third-order dispersion are present can be reasonably described in terms of a ''two-stage'' perturbation theory. In contrast to the adiabatic soliton perturbation theory which is known to fail beyond a critical strength of third-order dispersion, a modulation of the soliton phase caused by third-order dispersion is taken into account in this approach. [S1063-651X(97)07302-9]

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Among others, there are two fundamental problems in using solitons in optical fiber communication lines: to reduce the fiber dispersion (however, keeping it anomalous), in order to allow the weak Kerr nonlinearity to produce a sufficiently narrow soliton, and to compensate the dissipative losses in long-haul systems [1], simultaneously suppressing the noise. In order to cope with the former problem, a carrier wavelength near the zero-dispersion point (ZDP) can be used, which in addition leads to the reduction of the soliton peak power. The latter problem can be resolved by means of bandwidth-limited amplification (BLA), i.e., the combined action of optical amplifiers and filters.

A complicating factor is that third-order dispersion (TOD) has usually to be taken into account near the ZDP. Although TOD is a Hamiltonian perturbation it gives rise to the emission of radiation where the so-called resonance radiation separates from the soliton and can entail a complete destruction of the soliton [2–5]. Moreover, TOD leads to the breakup of the two-soliton bound state [2,6–8]. But it is also known that other perturbations can be exploited to suppress these detrimental effects of TOD. It was shown in [9] that, e.g., the decay of the two-soliton bound state can be avoided if BLA comes into play. Moreover, it has been predicted [8,10] and experimentally verified lately [11] that BLA may absorb the emitted resonance radiation, thus lending the soliton a much better stability. Thus in view of these observations it is necessary to analyze the dynamical properties of solitons in the presence of several qualitatively different perturbations.

The effect of TOD on the dynamical behavior of a single soliton has been intensively studied [12,13,6,14,4,5]. The soliton velocity and frequency shifts which appear due to the presence of TOD can correctly be described by transforming the pertinent perturbed nonlinear Schrödinger equation (NLSE) into a perturbed, but integrable equation of the NLSE family which is correct up to the second order in the TOD coefficient [13,6]. The frequently used adiabatic approximation of the soliton perturbation theory for the NLSE (AST) (see [15–17] and [18] for a review) describes the soliton velocity in first order of the TOD coefficient, but fails to provide the frequency shift [1,8]. Only recently it has been shown that this frequency shift can also be obtained in this model provided that the constraint of the adiabatic evolution is lifted [5]. This means that the mutual interaction of the nonresonant radiation and the soliton, both propagating with the same velocity, has been taken into account [5]. For the sake of simplicity we use the term ''velocity'' although actually an inverse velocity is meant.

The effect of TOD as well as that of TOD in combination with BLA on the two-soliton interaction has been recently studied [8,19] by using the AST. Provided that a weak TOD was the only perturbation, a good agreement with results obtained from direct numerical simulations has been found [8]. However, for increased, but still realistic values of TOD, the AST has been shown to fail. In particular, the velocity imposed on the soliton by TOD may differ by a factor of 2 if AST and numerical methods are compared [8]. If only TOD acts as a perturbation this discrepancy between the numerical and an analytical approach can be reduced in using the model of Kodama *et al.* [9]. Unfortunately, this model cannot be applied to non-Hamiltonian perturbations such as, e.g., BLA and nonlinear loss or gain (NLG). Basically, there are two options to explain the failure of AST, viz., either the emission of resonance radiation [3] or the nonadiabatic behavior of the soliton. One is inclined to prefer the latter explanation because BLA absorbs the resonance radiation [8,11], as already mentioned above.

Recently, the soliton propagation was investigated under the combined action of TOD (introduced by the filter), BLA, and sliding [20]. As an interesting result an asymmetry for up and down sliding was found with respect to the gain required to compensate for the losses. The basic idea of the perturbation approach used was to distinguish between two groups of perturbations and to treat them consecutively. In the first stage a phase modulation (chirp) induced by TOD has been taken into account. Having plugged this ansatz into the evolution equations for the energy and the momentum, the dynamics of the soliton amplitude and mean frequency due to BLA and sliding filtering has been studied. The results obtained were in good agreement with direct numerical

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simulations. This is in some respect surprising because the above separation might suggest that TOD has to be the *strongest* perturbation whereas the authors analyzed just the opposite case [20]. Moreover, it has been shown previously [8] that the usual AST holds for the strength of TOD assumed in that paper. Hence the natural question arises whether this ''two-stage'' perturbation theory can reproduce the numerical results which were obtained earlier for a fairly strong TOD [8] and were in clear disagreement with the AST.

The aim of the present paper is therefore to disclose the potential of this two-stage approach for the case where TOD is the *primary* perturbation and linear BLA as well as nonlinear loss or gain are considerably weaker. Moreover, it is intended to identify the limits of applicability of AST as far as an increasing TOD is concerned.

We mention that recently a similar two-stage perturbation strategy has proven to be quite effective in another problem related to optical solitons, namely, the search for a soliton in a model of a dual-core fiber, with BLA in one core and pure losses in another [21]. In this case, the coupling between the two cores was taken into account first as a relatively strong perturbation. This has allowed construction of a corresponding two-component soliton, for which the gain and losses were next considered as the small perturbations. This approach has allowed analytical prediction of the existence of a stable soliton in the model. Very recently, these analytical results have been corroborated by direct simulations [22].

Because of the prominent role TOD plays in our case we first take into account a modulation of the phase induced by TOD, and then apply to the modified (chirped) soliton the familiar techniques [18] which allow us to handle effects of linear or nonlinear gain and losses and filtering. However, it will turn out that this procedure can even be applied if TOD and the remaining perturbations are comparable. This means that a strong TOD is a sufficient, but not a necessary condition for the applicability of the two-stage model. We note that the model used here resembles that put forward lately and referred to above [20] but differs in details. Moreover, in the present paper a completely different issue is addressed.

The model combining TOD, BLA, and NLG is based on the following perturbed averaged nonlinear Schrödinger equation for the envelope $u(z,\tau)$ of the electromagnetic waves in the fiber, where, as usual, z stands for the propagation distance and τ is the so-called reduced time [9,8]:

$$iu_z+\tfrac{1}{2}u_{\tau\tau}+|u|^2u=i\epsilon u_{\tau\tau\tau}+i\gamma_0 u+i\gamma_1 u_{\tau\tau}-i\gamma_2|u|^2u. \tag{1}$$

Here ϵ is the TOD coefficient and γ_0 is the net gain, while the coefficient γ_1 accounts for the linear losses due to the finite amplifier or filter bandwidth. Furthermore, we have taken into account nonlinear losses or gain, the strength of which is described by γ_2. Two-photon absorption corresponds to $\gamma_2>0$ whereas saturable absorption or nonlinear amplification in a loop mirror corresponds to $\gamma_2<0$.

We expect that the soliton will depend on $T\equiv\tau-cz$, c being the unknown inverse equilibrium velocity of the soliton. To obtain a solution of this form it is natural to transform Eq. (1) into the coordinates (z,T). This transformation will, obviously, produce the group velocity term $-icu_T$. To eliminate this term, we introduce the new field variable

$$u(z,\tau)\equiv U(z,T)\exp(ic\tau-\tfrac{1}{2}c^2z). \tag{2}$$

In the case of the usual NLS equation, this is tantamount to the Galilean transform.

The transformed equation will receive many additional terms. However, we expect that the velocity c, to be produced by the perturbation, will eventually have a certain smallness, which will indeed be justified by the final result. This fact allows one to omit all terms in the transformed equation proportional to different powers of c, except for the single one, $-2c\gamma_1U_T$, which is produced by insertion of Eq. (2) into the dispersive lossy term in Eq. (1). While all the other small terms will produce only small corrections to effects accounted for by the larger terms, this one will give rise to a new effect: a driving force which forces the soliton to move with a finite velocity c. Thus the final form of the perturbed NLS to be considered in the present work is

$$iU_z+\tfrac{1}{2}U_{TT}+|U|^2U=i\epsilon U_{TTT}+i\gamma_0U+i\gamma_1U_{TT}-2c\gamma_1U_T -i\gamma_2|U|^2U. \tag{3}$$

Notice that, actually, the net gain γ_0 in Eq. (3) must be replaced by $\gamma_0-\gamma_1c^2$, but, according to what was said above, we neglect this change.

Now, at the first stage of the perturbation theory, we completely neglect gain and losses and consider only TOD and assume that the soliton will have the form

$$U(z,T)=a(T)\exp[iqz+i\phi(T)], \tag{4}$$

where q is the propagation constant.

Our aim is to find the modulation of the soliton's phase (i.e., the local chirp) generated by this perturbation. Inserting Eq. (4) into Eq. (3), we obtain, in the lowest-order approximation, the following equation:

$$(a^2\phi_T)_T=2\epsilon a a_{TTT},$$

which can be immediately integrated:

$$a^2\phi_T=2\epsilon(aa_{TT}-\tfrac{1}{2}a_T^2). \tag{5}$$

As the zeroth-order approximation, the unperturbed profile of the soliton's amplitude:

$$a(T)=\eta\ \mathrm{sech}(\eta T), \tag{6}$$

where η^2 is the soliton's peak power can be used.

Hence taking into account only TOD the solution for the envelope is written now as (see also [20])

$$u(z,\tau)=\eta\ \mathrm{sech}[\eta(\tau-cz)] \times\exp\left[i\left(\frac{\eta^2}{2}z-\Omega\tau-3\epsilon\eta\tanh[\eta(\tau-cz)]\right)\right], \tag{7}$$

with $\Omega=-(\epsilon\eta^2+c)$. Evidently, the solution (7) is chirped. Hence the mean frequency ω differs from Ω. As usual it is defined by the ratio of the momentum and the energy which gives $\omega=\epsilon\eta^2-c=\Omega+2\epsilon\eta^2$. This result reflects that (in first order of TOD and c) the soliton can exhibit any, but of course small, velocity in dependence on its frequency ω. It can even rest provided that the initial pulse has the frequency $\omega = \epsilon\eta^2$. The chirp does not depend on the propagation distance if the soliton rests. Now it is clear that there is no unique equilibrium velocity c if only TOD acts. But one can expect that BLA selects from this spectrum a particular velocity due to its preference for definite frequencies.

The next step of the perturbation theory is to take into account the gain and losses, and to find then an equilibrium soliton solution. Actually, the perturbation theory must yield the values of the two unknown parameters η and c that correspond to the equilibrium in the presence of gain and losses. The simplest way to do this is to make use of the so-called balance equations [18] for the soliton's energy $E \equiv \int_{-\infty}^{+\infty} |U(T)|^2 dT$ and momentum $P \equiv i \int_{-\infty}^{+\infty} U_T^* U dT$. In the absence of the gain and dissipation, they are obvious integrals of motion. To obtain the equilibrium solution in the presence of these small perturbations, one should demand that this conservation is maintained with respect to varying z. To perform the actual calculation, the full perturbed NLSE, Eq. (3), is used and the expressions (6) for $a(T)$ and (5) for ϕ_T are eventually inserted.

The balance equation for the energy leads, in the lowest approximation, to the well-known result for the amplitude [18],

$$\eta^2 = 3\gamma_0(\gamma_1 + 2\gamma_2)^{-1}. \tag{8}$$

In this expression, we completely neglect all the additional contributions related to the velocity c. We have calculated those corrections too. However, for the numerical values of the parameters used below, they have changed the final result by no more than 1%.

The calculation of the momentum balance is a nontrivial element of the present approximation. We find

$$c = \epsilon\left(-\tfrac{3}{2}\gamma_0/\gamma_1 + \tfrac{27}{10}\eta^2 + \tfrac{7}{5}\eta^2\gamma_2/\gamma_1\right). \tag{9}$$

Now, it remains to insert the expression (8) into Eq. (9) to obtain the eventual result

$$c = \frac{3}{5}\,\epsilon\,\frac{\gamma_0}{\gamma_1}\,\frac{(11\gamma_1 + 2\gamma_2)}{(\gamma_1 + 2\gamma_2)}. \tag{10}$$

Setting $\gamma_2 = 0$, we obtain from Eq. (9) $c = \epsilon\frac{33}{5}\gamma_0/\gamma_1$ [23].

Thus we have the analytical prediction for the velocity at which the soliton is expected to move in the equilibrium state under the combined action of TOD, BLA, and NLG. The comparison of this result (10) with the outcome of the AST [8] and the data obtained by direct numerical integration of Eq. (1) in using the beam-propagation method (BPM) is presented in Fig. 1 and Table I. It is evident from Fig. 1 that AST only yields reliable results if TOD is weak ($\epsilon < 0.075$) whereas for strong TOD ($\epsilon = 0.2$) and for the combined action of TOD and BLA it fails completely. This can be explained by the fact that in both cases the pulses acquire a considerable chirp upon propagation (note the large propagation distance) which is not accounted for in the conventional AST. The primary result we can read off from Fig. 1 is that the two-stage perturbation model provides reasonable results as far as TOD and BLA act jointly. We note, however, that the second stage of the approach cannot be implemented if no BLA is present. The reason is that it essentially exploits the existence of the equilibrium soliton velocity which can be derived only in the presence of all (including dissipative) perturbations.

Finally, in Table I the analytical results (10) are compared with the numerical findings for different values of TOD,

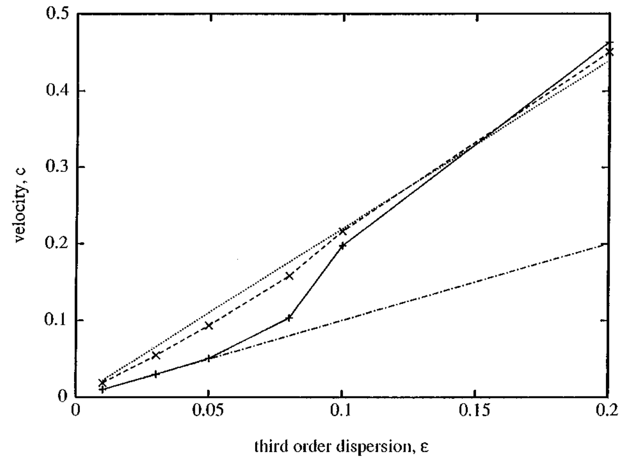

FIG. 1. Inverse equilibrium velocity c vs strength of third-order dispersion provided by different methods. BLA perturbation: $\gamma_0=0.01$ and $\gamma_1=0.03$; dashed line: BPM (distance of propagation—50 soliton periods); dotted line: two-stage model—Eq. (10); dash-dotted line: AST. For completeness we have added the BPM results without BLA (solid line). Note that AST provides the same results for the cases with and without BLA.

TABLE I. Comparison of the inverse equilibrium velocity calculated by BPM and the two-stage perturbation theory [Eq. (10)]. Note that the case $\epsilon=0.2$, $\gamma_0=0.05$, and $\gamma_1=0.15$ corresponds to Fig. 14 of Ref. [8] and this with $\epsilon=0.2$, $\gamma_0=0.02$, $\gamma_1=0.15$, and $\gamma_2=0.045$ to Fig. 4 of Ref. [9].

ϵ	γ_0	γ_1	γ_2	c_{an}	c_{num}	Err (%)
0.1	0.01	0.03	0	0.22	0.216	2
0.1	0.03	0.09	0	0.22	0.216	2
0.1	0.05	0.15	0	0.22	0.216	2
0.2	0.01	0.03	0	0.44	0.45	2
0.2	0.03	0.09	0	0.44	0.422	4
0.2	0.05	0.15	0	0.39	0.44	10
0.2	0.05	0.15	0.05	0.28	0.26	8
0.2	0.05	0.15	0.2	0.15	0.145	3
0.2	0.02	0.15	0.045	0.44	0.466	6

BLA, and NLG. Here $c_{\rm an}$ and $c_{\rm num}$ stand, respectively, for the analytical prediction (10) and numerical value, and ''Err'' is $|c_{\rm an}-c_{\rm num}|/c_{\rm num}$ (in percent). (Note that the cases considered earlier in [8,9] are included there.) The agreement between the analytical and numerical results is surprisingly good even when the BLA parameters γ_0 and γ_1 are of the same order of magnitude as the TOD coefficient ϵ, which indicates that the two-stage perturbation theory holds also for this case. The analytical results become less accurate for very strong TOD and comparable BLA, but the agreement is still quite satisfactory.

In conclusion, we have used a two-stage perturbation approach to analytically derive the equilibrium velocity solitons acquired in nonlinear optical fibers in the presence of bandwidth-limited amplification, and nonlinear gain or loss and *strong* third-order dispersion. Such a perturbation approach is a simple way to go beyond the adiabatic approximation of the soliton perturbation theory. We found that this approach rather than the conventional adiabatic soliton perturbation theory has to be used if TOD acts together with BLA. This holds even for a weak TOD provided that the propagation distance is large. The physical origin of this behavior is the chirp that the soliton acquires due to either effect and which is not accounted for in the conventional adiabatic perturbation theory. The results were found in reasonable agreement with recent numerical observations [9,8].

Finally we mention that the approach used here can be likewise applied if the Raman self-frequency shift and/or self-steepening are added to TOD, which is typically the case if femtosecond pulses are concerned. Moreover, it might be reasonable to consider the soliton-soliton interactions in the presence of TOD and BLA by applying this perturbation technique. A strong reduction of soliton interactions in the presence of BLA and TOD numerically observed [9,8] can be attributed to the acquired chirp. This interpretation of numerical studies is supported by the observation that chirped solitons (exact solutions of the Ginzburg-Landau equation with BLA and NLG) exhibit a considerably reduced interaction force [24]. Moreover, it is possible to predict, following the approach of [25], that all the two-soliton bound states will be destroyed when the velocity c exceeds a certain critical value. These issues will be considered in detail elsewhere.

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